УДК 621.384.633.6

# NONLINEAR DYNAMICS IN NUCLOTRON

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The paper represents an extensive study of the nonlinear beam dymanics in the Nuclotron. Chromatic effects, including the dependence of the betatron tunes on the amplitude, and chromatic perturbations have been investigated taking into account the measured field imperfections. Beam distortion, smear, dynamic aperture and nonlinear acceptance have been calculated for different particle energies and betatron tunes.

The investigation has been performed at the Laboratory of High Energies, JINR.

## Нелинейная динамика в нуклотроне

***Д.Динев***

Представлено систематическое исследование нелинейной динамики в нуклотроне. Хроматические эффекты, в том числе зависимость бетатронных частот от амплитуды, и хроматические пертурбации исследованы, учитывая вклад погрешностей магнитного поля. Деформация огибающей пучка, smear, динамическая апертура и нелинейный аксептанс вычислены в зависимости от энергии частиц и бетатронных частот.

Работа выполнена в Лаборатории высоких энергий ОИЯИ.

## 1. Introduction

In a real circular accelerator the focusing magnetic fields in the quadrupole lenses and the bending magnetic fields in the dipole magnets are far from being perfectly linear. There exist high order multipole components of these fields arising from fabrication tolerances.

In superconducting magnets the strongest nonlinear field distortions are due to persistent eddy currents in the magnet filaments. In dipole magnets persistent currents excite all multipoles but the sextupole component is prevalent.

On the other hand, the accelerated beam is not monoenergetic. The accelerated particles cover a range of energies (momenta). In Nuclotron the relative momentum spread is $\Delta p/p = \pm 4\cdot 10^{-3}$ at injection energy and $\Delta p/p = \pm 8\cdot 10^{-4}$ at maximum energy. This fact has as a result different kind of chromatic effects — a shift of betatron tunes (the so-called chromaticity) with particle energy and beam envelope distortions (the so-called chromatic perturbations).

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Besides this, a dependence of the betatron tunes on the amplitude of oscillations appears in multipolar magnetic fields.

In addition to the random and systematic field errors sextupole lenses with purely nonlinear magnetic fields are intently placed in any circular accelerator to control the chromaticity.

In superconducting accelerators the sextupole correctors are of less importance for the nonlinear character of the particle motion. The superconducting accelerators are said to be «error dominated» machines what means that the systematic and random field errors in the magnets are the most important source of nonlinearities.

The paper represents an extensive study of the nonlinear dynamics in the Nuclotron. The data from measurement of magnetic field imperfections in dipoles and quadrupoles up to 6th order (systematic and random; normal and skew) have been used for this investigation [2].

## 2. Chromatic Effects

Even in perfectly linear accelerator the parameters of the particle motion depend on the energy (momentum). Between these dependences the most important is the dependence of the frequency of betatron oscillations on momentum, the so-called natural chromaticity [3]. The natural chromaticity is defined as:

$$Q' = \frac{dQ}{d\delta}, \qquad \delta = \frac{dp}{p}, \tag{1}$$

$Q$ being the betatron tune.

We calculated the natural chromaticity in the Nuclotron taking into account the influence of the accelerator dipoles. For small machines with relatively small bending radius $\rho$ (or the same magnitude as the dispersion $D_x$) this influence is essential (up to 100%),

$$\frac{\partial Q_x}{\partial \delta} = -\frac{1}{4\pi B\rho} \int\limits_{\substack{\text{Quad.}\\ +\text{Dip}}} \beta_x \frac{\partial B_z}{\partial x}\, ds +$$

$$+\frac{1}{4\pi} \int\limits_{\text{Dip.}} \left( 2\frac{\beta_x}{B\rho}\frac{D}{\rho}\frac{\partial B_z}{\partial x} + \gamma_x \frac{D}{\rho} - 2\alpha_x \frac{D'}{\rho} - \frac{\beta_x}{\rho^2} - \frac{\beta_x}{B\rho} D \frac{\partial^2 B_z}{\partial x^2} \right) ds,$$

$$\frac{\partial Q_z}{\partial \delta} = \frac{1}{4\pi B\rho} \int\limits_{\substack{\text{Quad.}\\ +\text{Dip.}}} \beta_z \frac{\partial B_z}{\partial x}\, ds +$$

$$+\frac{1}{4\pi} \int\limits_{\text{Dip.}} \left( -\frac{\beta_z}{B\rho}\frac{D}{\rho}\frac{\partial B_z}{\partial x} + \gamma_z \frac{D}{\rho} - \frac{\beta_z}{B\rho} D \frac{\partial^2 B_z}{\partial x^2} \right) ds. \tag{2}$$

We calculated $Q_x'$ and $Q_y'$ by numerically integrating (2). The obtained values are given in Table 1.

**Table 1**

| Parameter | Value |
|---|---|
| Natural chromaticity | |
| $Q'_x$ | − 7.735 |
| $Q'_y$ | − 7.937 |
| Chromaticity at $B\rho$ = 1.0 Tm (systematic errors in dipoles) | |
| $Q'_x$ | − 10.206 |
| $Q'_y$ | − 5.346 |
| Chromaticity at $B\rho$ = 45.83 Tm (systematic errors in dipoles) | |
| $Q'_x$ | 4.889 |
| $Q'_y$ | − 22.398 |

There are two main negative effects of the chromaticity:

a) A spread of the betatron tunes $\Delta Q$ appears and as a result the beam occupies an area in the tune diagram instead of a point. The particles with energy different from the energy of the equilibrium particle fall in the resonance stopbands and are lost.

b) For bunched beams the transverse head-tail instability is developed.

As at injection ($B\rho$ = 1.0 Tm) the relative momentum spread is $\delta = \pm 4\cdot 10^{-3}$, the corresponding tune spread is:

$$\Delta Q_x = \pm 0.04, \qquad \Delta Q_y = \pm 0.02.$$

At maximum energy ($B\rho$ = 45.83 Tm) the relative momentum spread is $\delta = \pm 8\cdot 10^{-4}$ and therefore the spread in the betatron tunes is:

$$\Delta Q_x = \pm 0.004, \qquad \Delta Q_y = \pm 0.018.$$

We must underline that the criteria for linearity of the accelerator adopted after a large number of theoretical and experimental investigations on a big number of machines is $\Delta Q < 0.005$. Therefore the chromaticity in the Nuclotron must be corrected.

Two families of sextupole lenses are available in the Nuclotron for correction of chromaticity. They are placed just before the focusing and defocusing quadrupoles in the strait sections of each superperiod.

The tune shift produced by these sextupole lenses is given by:

$$Q'_{x,y} = \pm \frac{1}{4\pi} \int_0^l \beta_{x,y} K' D_x ds, \tag{3}$$

where $K' = \frac{1}{B\rho} \frac{\partial^2 B_y}{\partial x^2}$ is the sextupole strength, $D_x$ is the dispersion and $L$ is the accelerator circumference.

**Table 2**

| $B''$ | Value, $T/m^2$ |
|---|---|
| Without errors | |
| SF | 1.50 |
| SD | − 2.55 |
| With systematic errors in dipoles at $B\rho = 1.0$ Tm | |
| SF | 1.74 |
| SD | − 2.05 |
| With systematic errors in dipoles at $B\rho = 45.83$ Tm | |
| SF | 8.27 |
| SD | − 249.37 |

We have calculated numerically the strengths of the sextupole correctors from (2) taking into account the chromaticity levels in Table 1. They are given in Table 2.

In the presence of sextupole fields, a dependence of the betatron tunes on the amplitude of oscillations appears. The perturbation theory gives [4]:

$$\Delta Q_x = \frac{3\varepsilon_x}{64\pi} \sum_{m=1,3} \int_0^l K'(s)\, \beta_x^{3/2}(s)\, ds \int_0^s K'(s')\, \beta_x^{3/2}(s) \frac{\sin m(\mu(s') - \mu(s))}{m}\, ds. \tag{4}$$

Octupole fields also give such a dependence of the tunes on the amplitude. In this case the perturbation theory gives [5]:

$$\Delta Q_x = + \frac{1}{16\pi} \int_0^l K''(s)\, \beta_x(s) \left[ \frac{1}{2}\, \varepsilon_x\, \beta_x(s) - \varepsilon_y\, \beta_y(x) \right] ds, \tag{5}$$

$$\Delta Q_y = - \frac{1}{16\pi} \int_0^l K''(s)\, \beta_y(s) \left[ \varepsilon_x\, \beta_x(s) - \frac{1}{2}\, \varepsilon_y\, \beta_y(x) \right] ds. \tag{6}$$

We have calculated the dependence of betatron tunes in the Nuclotron on amplitude numerically integrating the above equations. The results obtained ($B\rho = 1.0$ Tm) are:

$$\frac{\Delta Q_x}{d\varepsilon_x} = 101.4, \quad \frac{\Delta Q_y}{d\varepsilon_y} = -12.4, \quad \frac{\Delta Q_x}{d\varepsilon_y} = \frac{\Delta Q_y}{d\varepsilon_x} = 4.4.$$

Taking into account that the emittances at $B\rho = 1.0$ Tm are $\varepsilon_x = \varepsilon_y = 30.10^{-6}\pi$ m we receive the following tune shifts:

$$\Delta Q_x = 3 \cdot 10^{-3}, \qquad \Delta Q_y = -\,0.3 \cdot 10^{-3}.$$

These are small values.

Another important chromatic effect are the chromatic perturbations. By chromatic perturbations we imply here the change of the linear optics functions beta and alpha with the energy.

Following Montague [6] we will describe the chromatic perturbations by the functions:

$$B = \frac{\beta(\delta) - \beta(0)}{\sqrt{\beta(\delta)\,\beta(0)}} \approx \frac{\Delta\beta}{\beta}, \tag{7}$$

$$A = \Delta\alpha - \alpha(0)\,\frac{\Delta\beta}{\beta}. \tag{8}$$

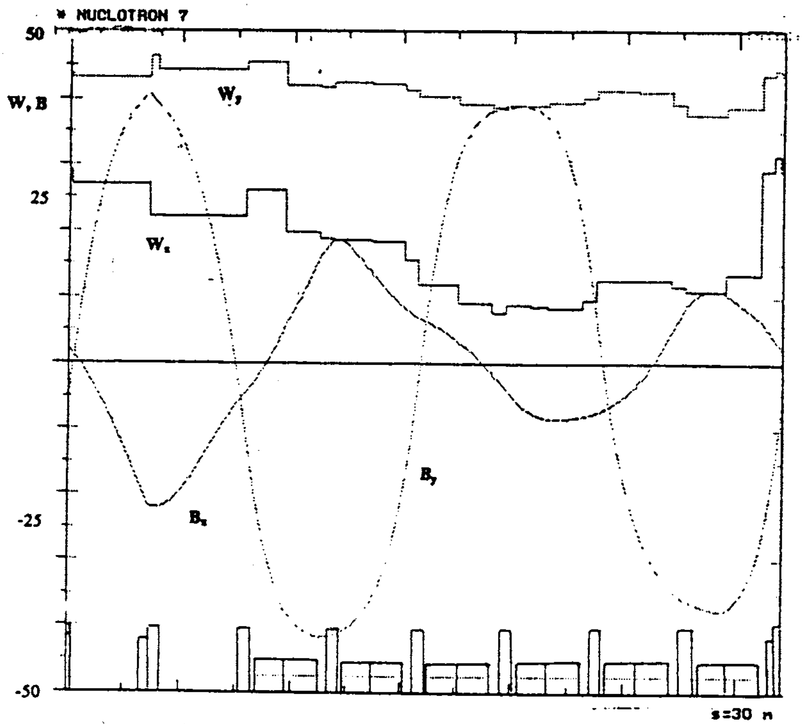
It can be shown that the vector

$$\mathbf{W} = (B, A) \tag{9}$$

rotates with a frequency $2Q$ in the achromatic areas.

In thin quadrupole

$$\Delta B = 0, \quad \Delta A = -\,\beta(KL)\,\delta. \tag{10}$$




**Montague chromatic functions (systematic errors, Bρ=45.83 Tm)**

Fig.1. Montague chromatic functions

**Table 3**

| Max. Montague chr. functions | Value |
|---|---|
| Without errors | |
| $W_{x,\, max}$ | 1.6δ |
| $W_{y,\, max}$ | 1.6δ |
| $B\rho$ = 1.0 Tm, systematic errors in dipoles | |
| $W_{x,\, max}$ | 13.1δ |
| $W_{y,\, max}$ | 9.8δ |
| $B\rho$ = 45.83 Tm, systematic errors in dipoles | |
| $W_{x,\, max}$ | 25.8δ |
| $W_{y,\, max}$ | 42.3δ |

In thin sextupole

$$\Delta B = 0, \quad \Delta A = -\beta(K'L)\,\delta. \tag{11}$$

We have calculated numerically the Montague functions using these properties. The calculated functions for maximum energy are plotted on Fig.1.

The maximum lengths of the vector $W$ taking into account the magnet imperfections are summarised in Table 3.

The maximum relative chromatic error in the amplitude function $\beta(s)$ taking into account the systematic errors in the dipoles at $B\rho$ = 1.0 Tm is:

$$\Delta\beta/\beta = \pm\, 6\%$$

and at $B\rho$ = 45.83 Tm:

$$\Delta\beta/\beta = \pm\, 4\%.$$

## 3. Beam Distortion

The Hamiltonian approach is the most convenient treatment of the nonlinear beam dynamics in circular accelerators. The Hamiltonian of a machine with nonlinearities can be written in the form [8]:

$$H(x, z, p_x, p_z) = H_0(x, z, p_x, p_z) + \sum_{N>2}^{\infty} H^{(N)}(x, z, p_x, p_z), \tag{12}$$

where $x$ and $z$ are transverse particle coordinates, $p_x$ and $p_z$ are the corresponding conjugate momenta, $\theta$ is the particle azimuth which is taken here as independent variable instead of the time, $H_0$ is the linear Hamiltonian and $H^{(N)}$ is the Hamiltonian of a nonlinearity of order $N$

$$H_0 = \frac{R^2}{2B\rho} \frac{\partial B_z}{\partial x} (x^2 - z^2) + \frac{1}{2} (p_x^2 + p_z^2). \tag{13}$$

In (13) $R$ is the mean radius of the accelerator, $B\rho$ is the beam rigidity which is determined by the equilibrium particle momentum through:

$$p = eB\rho, \tag{14}$$

$$H^{(N)} = \sum_{\substack{k_1, k_2 \\ k_1 + k_2 = N}} V_{k_1 k_2} x^{k_1} z^{k_2}, \tag{15}$$

$$V_{k_1 k_2} = \begin{cases} \dfrac{R^2}{B\rho N!} \begin{pmatrix} N \\ k_2 \end{pmatrix} (-1)^{\frac{k_2}{2}} \dfrac{\partial^{(N-1)} B_z}{\partial x^{(N-1)}}, & \text{for } k_2 \text{ even} \\ \dfrac{R^2}{B\rho N!} \begin{pmatrix} N \\ k_2 \end{pmatrix} (-1)^{\frac{k_2+1}{2}} \dfrac{\partial^{(N-1)} B_x}{\partial x^{(N-1)}}, & \text{for } k_2 \text{ odd.} \end{cases} \tag{16}$$

The corresponding Hamilton equations of motion are highly nonlinear in the general case and the transverse degrees of freedom are coupled. That is why analytical solutions exist only in some particular cases.

Several approximate analytical and numerical methods for treatment of the nonlinear beam dynamics in circular accelerators have been developed and successfully applied in practice. We have used two of them, namely the T.Collins analytical method of distortion fuctions and the numercial method of particle tracking.

In a perfectly linear accelerator the particles occupy the volume of a 6-dimensional ellipsoid in the 6-dimensional single particle phase space. As in the linear case the motions in both the transverse directions $x$ and $z$ and in the longitudinal direction $s$ are decoupled, the 6-dimensional phase space is transformed to three independent phase planes: $(x, p_x)$, $(z, p_z)$ and $(\theta, p_\theta)$, $\theta = s/R$ being the azimuth. The area occupied by the particles in each phase plane is confined by an ellipse. This ellipse is invariant of the motion (Courant–Snyder invariant) which means that if a particle lies on the ellipse after one turn, it will lie again on the same ellipse.

In the machine with nonlinearities the beam profile is not so simple as the horizontal and the vertical motions are coupled. The particles move on a distorted surface in four dimensional transverse phase space, the so-called «hyper-egg». We can talk only about the projections of this hyper-egg onto the two transverse phase planes $(x, p_x)$ and $(z, p_z)$. These projections are no longer clean curves but bands.

T.Collins [9] has shown that for sufficiently weak sextupole or octupole nonlinearities the beam distortion will be linear or quadratic function of the strengths of the nonlinearities.

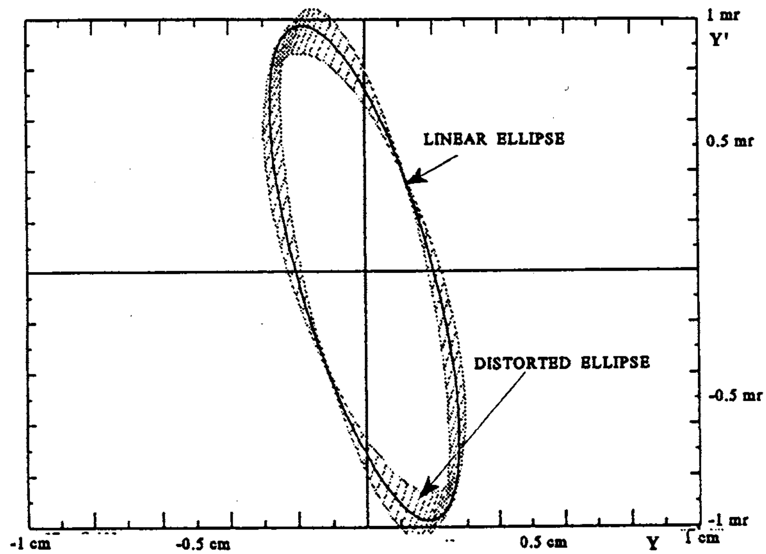



Fig.2. Beam distortion, $B\rho$ = 45.83 Tm

He devised a set of closed, i.e., periodic functions, the so-called distortion functions. These functions are independent of the beam amplitude and depend only on the linear lattice properties and the strengths of the nonlinearities. They are nonlinear analogues to the beta-functions and alpha-functions of the linear theory.

With the help of the distortion functions we can construct an invariant beam shape for the nonlinear case. This shape is mapped back in its turn if we use only terms of the first order in the sextupole and octupole strengths. If the second order terms are used,the beam shape will be distorted still further but the new invariant beam shape can be calculated through the second order distortion functions.

In this way a self-consistent result for the shape in a single particle phase space is developed order by order in the nonlinearities strengths just as in the perturbation theory.

T.Collins has shown that all the important effects of nonlinear fields can be derived from the set of distortion functions including the betatron tune shifts $\Delta Q_x$ and $\Delta Q_z$ with the amplitude of the oscillations.

We will not reproduce here the explicit expressions for the distortion functions (there are only five such functions for the normal sextupoles) but will turn our attention to a discussion of the results obtained for the Nuclotron applying this method.

Figure 2 shows the beam distortion at $B\rho$ = 45.83 Tm.

The beam envelope is not a clean curve but a band as this is the projection of the squashed hyper-egg in the four dimensional transverse phase space onto the phase plane. The width of the distortion figure is a measure for the deviation of the accelerator from the linear machine. The figure of merit is called SMEAR and is defined through:

$$\mathrm{SMEAR} = \frac{\sigma(A)}{\langle A \rangle}, \quad A = \sqrt{A_x^2 + A_z^2}, \tag{17}$$

where $A_x$ and $A_z$ are the horizontal and vertical amplitudes, $\sigma$ denotes the standard deviation and $\langle\ \rangle$ — the mean value.

For Nuclotron we have calculated that SMEAR = 5.3% at the working point $Q_x = 6.8$ and $Q_z = 6.85$. This is a rather small value. It was decided (SSC, LHC, HERA) that if SMEAR < 6.4%, the machine is considered to be sufficiently linear.

## 4. Dynamic Aperture

One of the most important characteristics of the circular accelerators is the dynamic aperture [10]. By definition the dynamic aperture is the area of the single particle phase space in which the particle motion is stable. It is a well-known fact that in presence of nonlinear fields the single particle phase space structure gets quite complicated and is divided to areas of stable and unstable motion.

It is very important for one to know how much of the phase space is stable. If this stable area is not large enough the normal operation of the machine might be destroyed.

In case a particle is injected into the accelerator with initial coordinates outside the dynamic aperture, the trajectory of that particle will be unstable and finally it will be lost on the vacuum chamber walls.

As it has been mentioned in chapter 1 no analytical solutions of the equations of motion in the general case of nonlinear magnetic fields have been found. That is why a quite straightforward numerical approach for calculation of dynamic aperture has been developed — the method of particle tracking [11].

The method consists of launching a particle into the accelerator with given initial coordinates and traking its motion for several hundreds turns (or even up to $10^6$ turns) in order to determine whether the particle trajectory is stable.

The problems arising from the nonlinearities are solved in the following way [11]. We consider the particle motion in dipoles and quadrupoles purely linear not taking into account the multipole components of the fields. The nonlinearities are simulated by attaching a nonlinear lens at the end of each dipole and quadrupole. These nonlinear lenses are described in «impulsive» or «kick» approximation which means that we consider the lengths of the lenses $L \to 0$ while the lenses strengths $B_n \to \infty$, but $Lb_n \to$ finite constant, or in other words we take the field distribution in the nonlinear lenses as a $\delta$-function of the longitudinal coordinate. In this approximation the transverse particle coordinates $x$, $z$ keep constant when we cross the lens while the particle slopes $x'$, $z'$ jump to new values — the so-called «kicks».

In the particle tracking treatment of the nonlinear problem we obtain the dynamic aperture as the maximum amplitude of a probe particle for which the trajectory is stable over a given number of turns. It has been noticed that the dynamic aperture defined that way depends rather strongly on the initial phase of the particle [12]. That is why many particles with different initial phases should be tracked in order to obtained the real dynamic aperture.

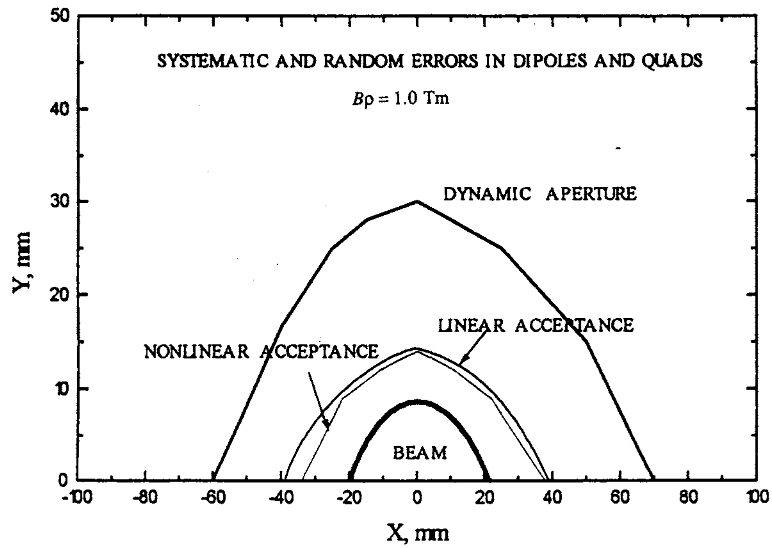



Fig.3. Dynamic aperture and nonlinear acceptance for $B\rho$ = 1.0 Tm

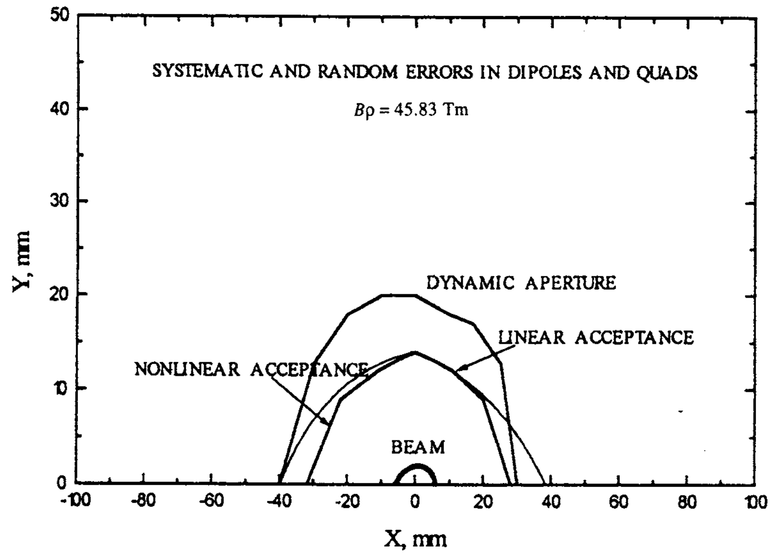



Fig.4. Dynamic aperture and nonlinear acceptance for $B\rho$ = 45.83 Tm

Due to the limited computer power the number of tracked turns in our calculations was set to 500, which is a commonly used value. The dynamic aperture calculated for such a low number of turns is referred as short-term dynamic aperture.

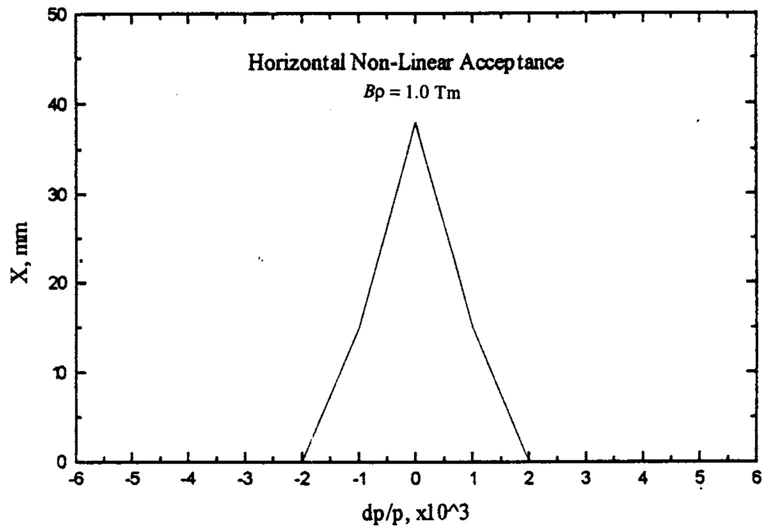



Fig.5. Dependence of the nonlinear acceptance on particle momentum

One should distinguish between the area of stable motion obtained with imposing of aperture limitation representing the real vacuum chamber sizes and without such limitations. We will call the area of stable motion «nonlinear acceptance» in the first case preserving the name «dynamic aperture» for the latter case when no real physical aperture but rather artificial limiting value is used. In our numerical calculations we use an amplitude limit of 1 meter for obtaining the dynamic aperture and aperture limits of 0.04 m in quadrupoles (radius) and of 0.056 m in dipoles (full poles gap) for obtaining the nonlinear acceptance.

Figure 3 shows the dynamic aperture and the nonlinear acceptance for the injection energy $E$ = 12 MeV/A ($B\rho$ = 1.0 Tm) while Fig.4 shows all these for the maximum beam energy $E$ = 6 GeV/A ($B\rho$ = 45.83 Tm).

Figure 5 depicts the dependence of the nonlinear acceptance on the particle momentum (energy).

## References

1. Vasilishin B.V. et al. — JINR Preprint 9-86-512, Dubna, 1986.
2. Donyagin A.M. — JINR Preprint P9-94-329, Dubna, 1994.
3. Verdier A. — Fouth Advances Accelerator Physics Course, CERN 02-01, Geneva, 1992.
4. Collins T. — AIP Conference Proceedings, 1983, v.105, p.92.
5. Autin B., Willeke F. — Second Advanced ICFA Beam Dynamics Workshop, CERN 88-04, Geneva, 1988.
6. Autin B. — AIP Conference Proceedings, 1987, v.153, p.288.

7. Montague B.W. — CERN Preprint, LEP Note 165, Geneva, 1979.
8. Guignard G. — CERN Preprint, ISR-MA 75-23, Geneva, 1975.
9. Collins T. — FNAL Preprint 84-114, Batavia, 1984.
10. Douglas D. — AIP Conference Proceedings, 1987, v.153, p.390.
11. Iselin F.Ch. — CERN Preprint, LEP-TH 85-26, Geneva, 1985.
12. Brinkmann R., Willeke F. — CERN Preprint 88-04, Geneva, 1988.

Received on December 10, 1997.